On the set of extreme core allocations for minimum cost spanning tree problems

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Abstract

Minimum cost spanning tree problems connect agents efficiently to a source when agents are located at different points and the cost of using an edge is fixed. We propose a method, based on the concept of marginal games, to generate all extreme points of the corresponding core. We show that three of the most famous solutions to share the cost of mcst problems, the Bird, folk and cycle-complete solutions, are closely related to our method.

Keywords: Minimum cost spanning tree problems; extreme core allocations, reduced game, Bird solution, folk solution, cycle-complete solution.

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1 Introduction

Minimum cost spanning tree (mcst) problems model a situation where agents are located at different points and need to be connected to a source in order to obtain a good or information. Agents do not care if they are connected

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directly to the source or indirectly through other agents who are. The cost to build a link between two agents or an agent and the source is a fixed number, meaning that the cost is the same whether one or ten agents use that particular link. Mcst problems can be used to model various real-life problems, from telephone and cable TV to water supply networks.

The core of mcst problems has been and early focus of attention, with Bird (1976) and Granot and Huberman (1981) showing that it is always non-empty and Granot and Huberman (1984) providing an algorithm to generate multiple core allocations. We present an improvement over these results by providing a method that allows to obtain the full set of extreme core allocations.

The method is based on the concept of marginal games (Núñez and Rafels (1998)), where we assign an agent her marginal cost to join the grand coalition, remove her from the problem and update the stand-alone costs of the remaining coalitions: they can either keep their original stand-alone cost or the stand-alone cost of them with the departing player, net of her cost share. This reduction is itself a special case of the Davis-Maschler reduction (Davis and Maschler (1965)). Given an ordering of the agents, repeating the process until all players are removed allows to find an extreme core allocation.

This method or very similar ones have been implemented for the assignment problem (Núñez and Rafels (2003)) and shortest path problems (Bahel and Trudeau (2014)), among others. In the non-cooperative setting, a similar approach consists in ordering buyers according to a given permutation and letting them buy goods in that order (Pérez-Castrillo and Sotomayor (2002), Vidal-Puga (2004))

The method does not work as well on all problems. Núñez and Rafels (1998) provide sufficient conditions for the method to always generate extreme core allocations. Under stricter conditions, we obtain the full set of extreme core allocations[1]. These conditions are not satisfied by mcst problems. We are still able to prove that the method generates the full set of extreme core allocations, using a representation of marginal games as minimum cost spanning tree problems with priced nodes. This new problem is a generalization of both mcst problems and Steiner tree problems (Hwang and Richards (1992), Skorin-Kapov (1995)).

[1] A related approach is that of (Tijs et al., 2011), which also looks for extreme core allocations given some lexicographic order. However, their approach is explicitely based on the core constraints and not on marginal games.

By taking the average of these extreme core allocations for all permutations, we obtain a very natural cost sharing solution, that corresponds to the barycenter of the core.[2] If the game is concave, it also corresponds to the Shapley value. We show that our procedure is very close to three well-known cost-sharing solutions for mcst problems.

First, if we only consider permutations that correspond to the order in which we connect agents in an optimal network configuration, we obtain directly the Bird solution (Bird (1976)). The Bird solution was the first solution to be shown to always be in the core and it is known for its simplicity, as we may assign cost at the same time as we construct an optimal tree.

Secondly, we show that for elementary problems (where all costs are either 0 or 1), our solution corresponds to the cycle-complete solution (Trudeau (2012)). The cycle-complete solution is obtained by modifying the cost of some links before taking the Shapley value of the corresponding cost game: we reduce the cost of edge (i, j) if there exists a cycle that goes through nodes i and j and such that its most expensive edge is cheaper than the direct edge (i, j). The modification is enough to make the corresponding cost game concave, and thus the Shapley value stable.

Thirdly, we show that for elementary problems, if we modify the game to make it monotonically increasing (so that an agent never reduces the cost when joining a coalition) before applying our solution, we obtain the folk solution (Feltkamp et al. (1994), Bergantiños and Vidal-Puga (2007)). The folk solution is obtained in the same way as the cycle-complete solution, but we look at paths instead of cycles. An interpretation of our result is that the folk solution is the barycenter of the non-negative core (the set of stable allocations such that no agent is subsidized).

Given that for the cycle-complete and folk solutions, we obtain cost shares for a general problem by decomposing it in a series of elementary problems, the results on their correspondence with our solution do not hold in general. However, our method provides a new way to extend the cycle-complete and folk solution from elementary problems to more general mcst problems.

The paper is divided as follows: Section 2 defines the minimum cost spanning tree problems. In Section 3 we describe our method and show that it generates extreme core allocations. We show that it generates the full set of extreme core allocations in Section 4. Links with popular cost sharing

[2] This is an abuse of language, as some permutations might yield identical extreme core allocations.

solutions are explored in Section 5. Section 6 contains some discussions.

2 The model

A (cost sharing) game is a pair (N, C) where $N = 1, \ldots, n$ is a nonempty, finite set of agents, and C is a characteristic function that assigns to each nonempty coalition $S \subseteq N$ a nonnegative cost $C(N) \in \mathbb{R}_+$ that represents the price agents in S should pay in order to receive a service. In particular, we assume that the agents in N need to be connected to a source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between nodes i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with $N_0^p = (N_0)^p$ and c_e representing the cost of edge e. Let Γ be the set of all cost vectors. Since c assigns cost to all edges e, we often abuse language and call c a cost matrix. A minimum cost spanning tree problem is a triple $(0, N, c)$. Since 0 and N do not change, we omit them in the following and simply identify a mcst problem $(0, N, c)$ by its cost matrix c.

A cycle p_{ll} is a set of $K \geq 3$ edges (i_k, i_{k+1}), with $k \in \{0, \ldots, K-1\}$ and such that $i_0 = i_K = l$ and $i_1, \ldots, i_{K-1}$ distinct and different than l. A path p_{lm} between l and m is a set of K edges (i_k, i_{k+1}), with $k \in \{0, \ldots, K-1\}$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $P_{lm}(N_0)$ be the set of all such paths between nodes l and m.

A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges. Its associated cost is $c(t) = \sum_{e \in t} c_e$.

We call mcst a spanning tree that has a minimal cost. Note that the mcst might not be unique. Let $t^*(c)$ be a mcst and $T^*(c)$ be the set of all mcst for the cost matrix c. Let $C(N, c)$ be the cost of a mcst. Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $C(S, c)$ be the cost of the mcst of the problem (S, c^S). Given these definitions, we say that C is the stand-alone cost function associated with c.

An allocation is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = C(N)$. For any $S \subseteq N$, let $x(S) = \sum_{i \in S} x_i$. Given $S \subseteq N$ and $x \in \mathbb{R}^N$, we denote as $x_S \in \mathbb{R}^S$ the restriction of x to $\mathbb{R}^S$.

For any cost matrix c, the associated cost game is given by (N, C) with $C(S) = C(S, c)$ for all $S \subseteq N$. We then say that C is a mcst game. We define the set of stable allocations as $Core(C)$. Formally, an allocation $x \in Core(C)$

if $x(S) \leq C(S)$ for all $S \subseteq N$.

3 A method to find extreme core allocations

We provide a method that allows to find extreme allocations of $Core(C)$. It is based on the concept of marginal games (Núñez and Rafels (1998)). We take an ordering of the agents and allocate to the last of them her marginal cost to join the grand coalition. We then update the stand-alone cost of remaining coalitions by giving them the option, instead of using their own stand-alone cost, to use their stand-alone cost with the removed player, net of her allocation. We then repeat the process by removing the next-to-last player, and so on.

Formally, for all $i \in N$ and all C, let $b_i^C = C(N) - C(N \setminus \{i\})$. For all $S \subseteq N$, the ith-marginal game $(N \setminus \{i\}, C^i)$ is given by $C^i(\emptyset) = 0$ and

$$C^i(S) = \min\left\{C\left(S \cup \{i\}\right) - b_i^C, C(S)\right\}$$

for all $\emptyset \neq S \subseteq N \setminus \{i\}$.

We label $C^{ij}(S)$ to mean $\left(C^i\right)^j(S)$, $C^{ijk}(S)$ to mean $\left(C^{ij}\right)^k(S)$ and so on.

Let $\Pi(N)$ be the set of permutations of N. Let $\pi = (\pi_1, \ldots, \pi_n) \in \Pi$. We define the reduced marginal cost vector of C related to permutation π, denoted as $y^{r\pi}(C)$, or simply $y^{r\pi}$, as follows:

$$\begin{aligned}
y_{\pi_n}^{r\pi} &= C(N) - C(N \setminus \{\pi_n\}) = b_{\pi_n}^C \\
y_{\pi_{n-1}}^{r\pi} &= C^{\pi_n}(N \setminus \{\pi_n\}) - C^{\pi_n}(N \setminus \{\pi_{n-1}, \pi_n\}) = b_{\pi_{n-1}}^{C^{\pi_n}} \\
&\vdots \\
y_{\pi_2}^{r\pi} &= C^{\pi_n \pi_{n-1} \ldots \pi_{n-3}}(\{\pi_1, \pi_2\}) - C^{\pi_n \pi_{n-1} \ldots \pi_{n-3}}(\pi_1) = b_{\pi_2}^{C^{\pi_n \pi_{n-1} \cdots \pi_{n-3}}} \\
y_{\pi_1}^{r\pi} &= C^{\pi_n \pi_{n-1} \ldots \pi_{n-2}}\left(\{\pi_1\}\right) = b_{\pi_1}^{C^{\pi_n \pi_{n-1} \cdots \pi_{n-2}}}.
\end{aligned}$$

Núñez and Rafels (1998) provide a sufficient condition for $y^{r\pi}$ to be an extreme point of the core, and in fact for the set $\left(y^{r\pi}\right)_{\pi \in \Pi}$ to be the set of extreme points of the core. The sufficient condition is that of almost-concavity of the cost game: $C(S) + C(T) \geq C(S \cup T) + C(S \cap T)$ for all $S, T \subset N$ such that $S \cup T \neq N$. We thus have all concavity conditions except those involving the grand coalition. We show with the following example that this condition is not satisfied by games generated by mcst problems.

Example 1 *Let* $N = \{1,2,3,4\}$ *and* c *be as described in the following (i horizontally, j vertically) and illustrated in Figure 1:*

c_{ij}	*1*	*2*	*3*	*4*
0	*1*	*6*	*5*	*5*
1		*6*	*4*	*2*
2			*5*	*5*
3				*5*

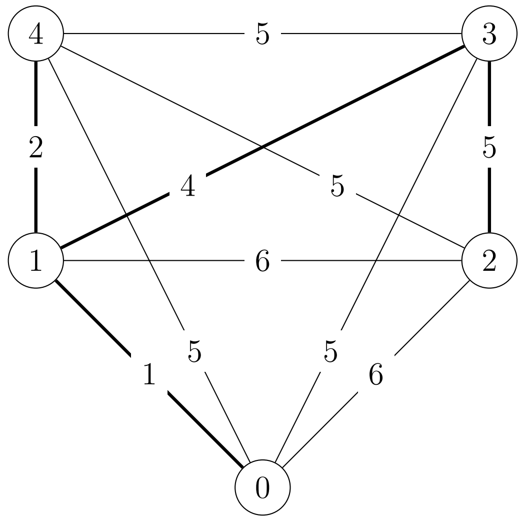

Figure 1: Example of a minimum cost spanning tree problem.

It is easy to see that $C(\{2\}) = 6$, $C(\{2,3\}) = C(\{2,4\}) = 10$ *and* $C(\{2,3,4\}) = 15$. *We thus have that* $C(\{2,3\}) + C(\{2,4\}) < C(\{2,3,4\}) + C(\{2\})$*, which contradicts the almost-concavity condition.*

It turns out that we can interpret a reduced game as a particular type of source connection problem. A *mcst problem with priced nodes* is a tupla (N, P, y, c) where $P \subseteq N$ are nodes that do not need to be connected and $y \in \mathbb{R}^P$ is the vector whose coordinates are the prices that nodes in P pay (or receive, when negative) if they are actually connected. Nodes in P are called *priced nodes*. Hence, the cost of (N, P, y, c) is defined as

$$C(N, P, y, c) = \min_{T \subseteq P} \{C((N \setminus P) \cup T) - y(T)\}$$

and the cost of a subset $S \subseteq N \setminus P$ is given by

$$C(S, P, y, c) = \min_{T \subseteq P} \{C(S \cup T) - y(T)\}.$$

In particular, $C(N,P,y,c) = C(N \setminus P, P, y, c)$.

As usual, the *core* of a mcst problem with priced nodes (N,P,y,c) is the set of allocations $x \in \mathbb{R}^{N \setminus P}$ that satisfy $x_i(N \setminus P) = C(N,P,y,c)$ and $x_i(S) \leq C(S,P,y,c)$ for all $S \subseteq N \setminus P$.

Notice that mcst problems with priced nodes generalize both mcst problems (when $P = \emptyset$) and minimum cost Steiner tree problems (when $y_i = 0$ for all $i \in P$).

For simplicity, when $y \in \mathbb{R}^N$, we write (N,P,y,c) instead of (N,P,y_P,c). Given $\pi \in \Pi(N)$ and $i \in N$, let $P^{\pi i} = \{\pi_i, \ldots, \pi_n\}$ be the set of nodes that come after π_i (including π_i) in the order π. It is clear that $C^{\pi_n \cdots \pi_i}(S) = C\left(S, P^{\pi i}, y^{r\pi}, c\right)$ for all $S \subseteq N \setminus P^{\pi i}$.

This correspondence allows us to prove that $y^{r\pi}$ belongs to the core.

Theorem 1 *For any mcst game (N,C) and any permutation $\pi \in \Pi(N)$, $y^{r\pi} \in Core(C)$.*

Proof. Given that $C^{\pi_n \cdots \pi_i}(S) = C\left(S, P^{\pi i}, y^{r\pi}, c\right)$ for all $S \subseteq N \setminus P^{\pi i}$, it suffices to prove that $y^{r\pi}_{N \setminus P^{\pi i}}$ belongs to the core of $(N, P^{\pi i}, y^{r\pi}, c)$.

Let $S \subseteq N \setminus P^{\pi i}$ and let t^S be a tree that connects all the nodes in S to the source using a set $T^S \subseteq P^{\pi i}$ of priced nodes. We need to prove

$$y^{r\pi}(S) \leq \sum_{e \in t^S} c_e - y^{r\pi}\left(T^S\right)$$

or, equivalently,

$$y^{r\pi}\left(S \cup T^S\right) \leq \sum_{e \in t^S} c_e. \tag{1}$$

Assume there exists $t^* \in T^*(c)$ that is also an optimal tree in $(N,N,y^{r\pi},c)$. Hence

$$\sum_{e \in t^*} c_e - y^{r\pi}(N) \leq \sum_{e \in t^S} c_e - y^{r\pi}\left(S \cup T^S\right). \tag{2}$$

Since $\sum_{e \in t^*} c_e = y^{r\pi}(N)$, we have that (1) and (2) coincide.

We still need to prove that there exists $t^* \in T^*(c)$ that is also an optimal tree in $(N,N,y^{r\pi},c)$. We will prove a stronger result: that *any* $t^* \in T^*(c)$ is an optimal tree in *any* $(N, P^{\pi i}, y^{r\pi}, c)$. Assume w.l.o.g. $\pi = 1 \ldots n$. For notational convenience, let $P^{\pi 0} = \emptyset$. We proceed by induction on i. For $i = 0$, the result holds trivially. Assume the result holds for $i - 1 \geq 0$. Let $t^* \in T^*(c)$. We have to prove that t^* is an optimal tree on $(N, P^{\pi i}, y^{r\pi}, c)$, i.e. for each tree t that connects all the nodes in $(N \setminus P^{\pi i}) \cup T$ to the source, with $T \subseteq P^{\pi i}$, it holds that

$$\sum_{e \in t^*} c_e - y^{r\pi}\left(P^{\pi i}\right) \leq \sum_{e \in t} c_e - y^{r\pi}(T). \tag{3}$$

By induction hypothesis, t^* is an optimal tree in $(N, P^{\pi i-1}, y^{r\pi}, c)$. Assume first $i-1 \in T$. Then, $\sum_{e\in t^*} c_e - y^{r\pi}\left(P^{\pi i-1}\right) \leq \sum_{e \in t} c_e - y^{r\pi}\left(T \cup \{i-1\}\right)$, which is equivalent to (3). Assume now $i-1 \notin T$. Hence

$$\begin{aligned}
\sum_{e \in t^*} c_e - y^{r\pi}\left(P^{\pi i}\right) &= \sum_{e \in t^*} c_e - y^{r\pi}\left(P^{\pi i-1}\right) + y^{r\pi}_{i-1} \\
&= C\left(N, P^{\pi i-1}, y^{r\pi}, c\right) + y^{r\pi}_{i-1} \\
&= C\left(N \setminus P^{\pi i-1}, P^{\pi i-1}, y^{r\pi}, c\right) + y^{r\pi}_{i-1}.
\end{aligned}$$

By definition, $y^{r\pi}_{i-1} = C\left(N \setminus P^{\pi i}, P^{\pi i}, y^{r\pi}, c\right) - C\left(N \setminus P^{\pi i-1}, P^{\pi i}, y^{r\pi}, c\right)$ and $C\left(N \setminus P^{\pi i-1}, P^{\pi i}, y^{r\pi}, c\right) = C\left(N \setminus P^{\pi i-1}, P^{\pi i-1}, y^{r\pi}, c\right)$. Hence the above expression equals $C\left(N \setminus P^{\pi i}, P^{\pi i-1}, y^{r\pi}, c\right)$ which, by definition, is less or equal than $\sum_{e\in t} c_e - y^{r\pi}(T)$. ■

4 The set of extreme core allocations

We now have shown that our procedure generates allocations in the core. By definition, they are also extreme core allocations, defined as follows:

Definition 1 *An allocation $y \in Core(C)$ is an extreme core allocation if there does not exist $y', y'' \in Core(C)$, $y' \neq y''$ and $\lambda \in (0,1)$ such that $y = \lambda y' + (1-\lambda) y''$.*

We define the set of extreme core allocations as $ExtCore(C)$. We next show that the core allocations defined in the previous section actually constitute the whole set of extreme core allocations. We proceed by first showing that for any coalition, we attain the maximal allocation compatible with the core. We need to prove two intermediary results before getting to the main result of this section.

First, we define the cost function $\widehat{C}$ and show that it has the same core as C and that a coalition S cannot receive more than $\widehat{C}(S)$ in any core allocation.

For all $c \in \Gamma$, let $(N, \widehat{C})$ be defined in the following way:

Step 0: $\widehat{C}(S) = C(S)$ for all S such that $|S| \geq n-1$.

Step k: $\widehat{C}(S) = \min \left\{ C(S), \min_{T \subset N \setminus S} \widehat{C}(S \cup T) + \widehat{C}(N \setminus T) - \widehat{C}(N) \right\}$ for all S such that $|S| = n-1-k$, for $k = 1, \ldots, n-2$.

Step $n-1$: $\widehat{C}(\emptyset) = 0$.

Hence, $\widehat{C}(N) = C(N)$ and

$$\widehat{C}(S) = \min_{T \subset N \setminus S} \left\{ \widehat{C}(S \cup T) + \widehat{C}(N \setminus T) - C(N) \right\}$$

for all $S \subset N$.

To compute the alternative stand-alone cost $\widehat{C}(S)$, we let S pick a partner T and compute the sum of the costs of S with T and $N \setminus T$, to which we substract the cost of the grand coalition.

Lemma 1 *For all $c \in \Gamma$, $Core(\widehat{C}) = Core(C)$.*

Proof. Given that $\widehat{C}(S) \leq C(S)$ for all $S \subset N$ and $\widehat{C}(N) = C(N)$, it is obvious that $Core(\widehat{C}) \subseteq Core(C)$. We show that $Core(C) \subseteq Core(\widehat{C})$. Suppose, on the contrary, that there exists $y \in Core(C)$ and $y \notin Core(\widehat{C})$. Then, there exists $S \subset N$ such that $\widehat{C}(S) < y(S) \leq C(S)$. Stated otherwise, there exists $\varnothing \neq T \subset N \setminus S$ such that

$$\widehat{C}(S) = \widehat{C}(S \cup T) + \widehat{C}(N \setminus T) - \widehat{C}(N) < y(S) \leq C(S).$$

Suppose first that $|S| = n-2$. Then,

$$\widehat{C}(S) = C(S \cup T) + C(N \setminus T) - C(N) < y(S) \leq C(S) \tag{4}$$

as $|S \cup T|, |N \setminus T| \geq n-1$. Since $y \in Core(C)$, we must have that $y(S) + y(T) + y(N \setminus (S \cup T)) = C(N)$. We have the following core conditions:

$$\begin{aligned} y(S) + y(T) &\leq C(S \cup T) \\ y(S) + y(N \setminus (S \cup T)) &\leq C(N \setminus T). \end{aligned}$$

Adding these two constraints, we obtain

$$\begin{aligned} 2y(S) + y(T) + y(N \setminus (S \cup T)) &\leq C(S \cup T) + C(N \setminus T) \\ y(S) &\leq C(S \cup T) + C(N \setminus T) - C(N) = \widehat{C}(S). \end{aligned}$$

Therefore, we have a contradiction with (4).

We have thus shown that for $y \in Core(C)$, we need $y(S) \leq \widehat{C}(S)$ if $|S| = n-2$. We can then use a recursive argument to show that if $y \in Core(C)$ implies that $y(S) \leq \widehat{C}(S)$ for all S such that $|S| \geq n-k$, it also implies that $y(S) \leq \widehat{C}(S)$ for all S such that $|S| = n-k-1$. Thus, $y(S) \leq \widehat{C}(S)$ for all $S \subseteq N$ is a necessary condition for $y \in Core(C)$ and $Core(C) \subseteq Core(\widehat{C})$. ■

Our next step is to show that for any S, there exists at least one permutation π such that $y^{r\pi}(S) = \widehat{C}(S)$. To do so, we once again use the mcst with priced nodes representation.

We know, from the proof of Theorem 1, that $y^{r\pi}_{N\setminus P^{\pi i}}$ belongs to the core of $(N, P^{\pi i}, y^{r\pi}, c)$ and, moreover, that *any* $t^* \in T^*(c)$ is an optimal tree in *any* $(N, P^{\pi i}, y^{r\pi}, c)$.

For simplicity, and since N is fixed, we write Π instead of $\Pi(N)$.

Let $t^* \in T^*(c)$. We denote $t^* = \{(i, i^*)\}_{i \in N}$, where i^* is the predecessor of node i in t^*, i.e. i^* is the adjacent node to node i in the (unique) path in t^* from node i to the source.

We then define $i \preceq_* j$ as the partial relation in N given by "i precedes j in t^*", that is, $i \preceq_* j$ iff $j \in F^{*i}$, where F^{*i} is the set of followers of node i in t^* (including node i).

Given $P \subseteq N$ and $\pi \in \Pi$, we say that π is t^*-compatible with P if the following two conditions hold:

External t^*-compatibility Nodes in $S = N \setminus P$ that follow nodes in P in t^* are first in π:

$$\left.\begin{array}{l} \pi_i \preceq_* \pi_j \\ \pi_i \in P \\ \pi_j \in S \end{array}\right\} \Longrightarrow j < i.$$

Internal t^*-compatibility Nodes inside P follow in π the partial order $\preceq_*$:

$$\left.\begin{array}{l} \pi_i \preceq_* \pi_j \\ \pi_i, \pi_j \in P \end{array}\right\} \Longrightarrow i \leq j.$$

It is clear that there always exists an order t^*-compatible with P. For example, given $P \subseteq N$, $S = N \setminus P$ and $\pi \in \Pi$, we define $\pi^{*P} \in \Pi$ inductively as follows: If $P = \emptyset$, then $\pi^{*P} = \pi$. If $P = \{i\}$, then $\pi^{*P} = [\pi_S, i]$ is the order whose first components are the nodes in S (following order π) and the last component is node i. Assume we have defined $\sigma^{*Q} \in \Pi$ for any $\sigma \in \Pi$ and $Q \subset N$ with $|Q| < |P|$. Let $p_{\pi 1} \in P$, or p_1 for short, be the first node in π satisfying $p_1 \in P$ and $p_1^* \notin P$, where p_1^* is the adjacent node to p_1 in the (unique) path in t^* from p_1 to the source. Then, we apply the induction to define $\pi^{*P} = \left[\pi_{N \setminus \{p_1\}}, p_1\right]^{*P \setminus \{p_1\}}$.

Example 2 *We revisit Example 1, with* $P = \{1, 3, 4\}$.

Take $t^* = \{(0,1), (1,4), (1,3), (3,2)\}$. *External* t^**-compatibility implies that nodes* 1 *and* 3 *should go after node* 2. *Internal* t^**-compatibility implies that nodes* 3 *and* 4 *should go after node* 1. *These two conditions together give two unique* t^**-compatible orders:* $[2, 1, 4, 3]$ *and* $[2, 1, 3, 4]$.

If we choose an initial π *in which* 3 *has preference over* 4, *we obtain* $\pi^{*P} = [2, 1, 3, 4]$. *Otherwise, we obtain* $\pi^{*P} = [2, 1, 4, 3]$. *In both cases,* $y^{r\pi^{*P}} = (0, 6, 4, 2)$.

The same happens if we take $t^* = \{(0,1), (1,3), (1,4), (4,2)\}$ *instead.*

We say that π is c-compatible with P if there exists some $t \in T^*(c)$ such that π is t-compatible with P.

In the previous example, $y^{r\pi}(\{2\}) = \widehat{C}(\{2\}) = 6$ for each π c-compatible with $\{1, 3, 4\}$. We now show that this holds in general.

Theorem 2 *Given* $S \subseteq N$, *we have* $y^{r\pi}(S) = \widehat{C}(S)$ *for all* $\pi \in \Pi$ *order* c*-compatible with* $N \setminus S$.

Proof. Fix $t^* = \{(i, i^*)\}_{i \in N} \in T^*(c)$. Given $S \subseteq N$ and $P = N \setminus S$, let $\pi \in \Pi$ be an order t^*-compatible with P. We prove that the following two statements hold:

(I) Either $y^{r\pi}(S) = C(S)$ or there exists $\emptyset \neq T \subset P$ such that

- **(I**a**)** $y^{r\pi^a}(T) = y^{r\pi}(T)$ for all $\pi^a \in \Pi$ order t^*-compatible with T, and
- **(I**b**)** $y^{r\pi^b}(P \setminus T) = y^{r\pi}(P \setminus T)$ for all $\pi^b \in \Pi$ order t^*-compatible with $P \setminus T$.

(II) $y^{r\pi}(S) = \widehat{C}(S)$.

We proceed by induction on $|P|$. For $P = \emptyset$, both statements hold trivially. Assume now both statements hold when $|P| < \alpha$ and suppose $|P| = \alpha$ for some $\alpha > 0$.

We first prove statement (I). Let $p_1 = \pi_s \in P$ be the first element in P according to π (that is, $i < s$ implies $\pi_i \notin P$). Hence, for all $\pi_i \in P \setminus \{p_1\}$, we have $s < i$. Under internal t^*-compatibility, we deduce $p_1 \notin F^{*i}$ for all $i \in P \setminus \{p_1\}$.

Let $S' = S \cup \{p_1\}$ and $P' = P \setminus \{p_1\}$. By applying t^*-compatibility, it is straightforward to check that

$$y_{p_1}^{r\pi} = C\left(S', P', y^{r\pi}, c\right) - C\left(S, P', y^{r\pi}, c\right). \tag{5}$$

Let t' be an optimal tree in $(N \setminus \{p_1\}, P', y^{r\pi}, c)$. That is:

$$\begin{aligned} S &\subseteq t'(N) \\ p_1 &\notin t'(N) \\ c(t') - y^{r\pi}(t'(N) \cap P') &= \min_{T' \subseteq P'} \{C(S \cup P') - y^{r\pi}(T')\} \end{aligned}$$

where $t'(N) \subseteq N \setminus \{p_1\}$ is the set of nodes that connect to the source through t'.

In case there are more than one tree satisfying the above conditions, we take t' such that $|t'(N)|$ is maximal among them.

Assume first $S = t'(N)$, which means $t'(N) \cap P' = \emptyset$ and $C(t') = C(S)$. Hence

$$\begin{aligned} y^{r\pi}(S) &= y^{r\pi}(S') - y_{p_1}^{r\pi} \\ &\overset{(5)}{=} y^{r\pi}(S') - C(S', P', y^{r\pi}, c) + C(S, P', y^{r\pi}, c) \\ &= y^{r\pi}(S') - y^{r\pi}(S') + c(t') = c\left(t'\right) = C(S) \end{aligned}$$

and so statement (I) holds.

Assume now $S \subset t'(N)$. Let $T' = t'(N) \setminus S$. It is straightforward to check that $\emptyset \neq T' \subset P$. We will prove that (I$a$) and (I$b$) hold with this T'. As a previous step, we need to prove the following Claim:

Claim A: For all $i \in N \setminus \{p_1\}$, $i^* \in S \cup T' \Rightarrow i \in S \cup T'$.

Assume, on the contrary, that there exists some $i \in N \setminus \{p_1\}$ such that $i^* \in S \cup T'$ and $i \notin S \cup T'$. That is, $i^* \in t'(N)$ and $i \notin t'(N)$.

Let $Q = \left\{ j \in N \setminus T' : \tau^*_{ij} \subseteq N \setminus T' \right\}$, where τ^*_{ij} is the (unique) path form node i to node j in t^*. Thus, Q is the set of followers of node i without leaving $P \setminus T'$. Hence, $Q \subseteq P \setminus T'$. Since $p_1 \notin F^{*j}$ for all $j \in P'$, we deduce that $p_1 \notin Q$. Let $R = \{j \in N : j^* \in Q\}$ be the set of nodes in S that immediately follow some node in Q (case $R = \emptyset$ is also possible). Denote $R = \{r_1, \ldots, r_k\}$. We define the set of edges $E = \{e_1, \ldots, e_k\} \subset N^p$ inductively as follows: Let $e_1 = (i_1, i'_1) \in N^p$ be the first edge in the (unique) path in t' from r_1 to the source such that $i_1 \in F^{*r_1}$ and $i'_1 \notin F^{*r_1}$. Let $e_2 = (i_2, i'_2) \in N^p$ be the first edge in the (unique) path in t' from r_2 to the source such that $i_2 \in F^{*r_1} \cup F^{*r_2}$ and $i'_2 \notin F^{*r_1} \cup F^{*r_2}$. Let $e_3 = (i_3, i'_3) \in N^p$ be the first edge in the (unique) path in t' from r_3 to the source such that $i_3 \in F^{*r_1} \cup F^{*r_2} \cup F^{*r_3}$ and $i'_3 \notin F^{*r_1} \cup F^{*r_2} \cup F^{*r_3}$. And so on. Thus, E is the set of edges that connect R to the source in t'. Now,

$$t^1 = (t^* \setminus \{(j, j^*)\}_{j \in Q \cup R}) \cup E \tag{6}$$

is a tree that connects all the nodes in S to the source using nodes in $P \setminus Q$. Since t^* is optimal in $(S, P, y^{r\pi}, c)$, we deduce that

$$c(t^*) - y^{r\pi}(P) \leq c(t^1) - y^{r\pi}(P \setminus Q)$$

or, equivalently,

$$c(t^*) \leq c(t^1) + y^{r\pi}(Q). \tag{7}$$

Now,

$$t^2 = (t' \setminus E) \cup \{(j, j^*)\}_{j \in Q \cup R} \tag{8}$$

is a tree that connects all the nodes in S to the source using nodes in $T' \cup Q \subseteq P'$. Since t' is optimal in $(S, P', y^{r\pi}, c)$, we deduce that

$$c(t') - y^{r\pi}(T') \leq c(t^2) - y^{r\pi}(T' \cup Q)$$

or, equivalently,

$$c(t') \leq c(t^2) - y^{r\pi}(Q). \tag{9}$$

By applying (7) and (9),

$$\begin{aligned} c\left(t^2\right) &\overset{(9)}{\geq} c(t') + y^{r\pi}(Q) \overset{(7)}{\geq} c(t') + c(t^*) - c(t^1) \\ &\overset{(6)}{=} c(t') + c(t^*) - c(t^*) + c(\{(j, j^*)\}_{j \in Q \cup R}) - c(E) \\ &= c\left(t'\right) + c\left(\{(j, j^*)\}_{j \in Q \cup R}\right) - c\left(E\right) \\ &\overset{(8)}{=} c\left(t^2\right). \end{aligned}$$

This implies $c\left(t^2\right) = c\left(t'\right) + y^{r\pi}(Q)$, so

$$c\left(t^2\right) - y^{r\pi}(T' \cup Q) = c\left(t'\right) - y^{r\pi}(T')$$

and hence t^2 is also an optimal tree in $(S, P', y^{r\pi}, c)$ with $|t^2(N)| = |t'(N)| + |Q|$, which contradicts that $|t'(N)|$ is maximum among these optimal trees (notice that $i \in Q \neq \emptyset$). This contradiction completes the proof of Claim A.

We can now prove that (Ia) holds with T'. Under the induction hypothesis on statement (II), we have $y^{r\pi^a}\left(N \setminus T'\right) = \widehat{C}(N \setminus T')$ for all $\pi^a \in \Pi$ order t^*-compatible with T'. On the other hand, if π is t^*-compatible with T', then we can also apply the induction hypothesis on (II) to deduce that $y^{r\pi}\left(N \setminus T'\right) = \widehat{C}(N \setminus T')$. Hence, it is enough to prove that π is indeed t^*-compatible with T'. We check both external and internal t^*-compatibility.

External t^*-compatibility: Let i, j such that $\pi_i \preceq_* \pi_j$, $\pi_i \in T'$ and $\pi_j \in N \setminus T'$. We have three cases: If $\pi_j \in S$, then $j < i$ because π is externally t^*-compatible with P. If $\pi_j = p_1$, then $\pi_i \preceq_* p_1$ implies $s < i$ because $\pi_i \in T' \subset P$ and $p_1 = \pi_s$ is the first element of P in π; by internal t^*-compatibility with P, we deduce $\pi_i \not\preceq_* \pi_s = \pi_j$, which is a contradiction. Finally, if $\pi_j \in P \setminus (T' \cup \{p_1\})$, then $\pi_j \notin S \cup T' \cup \{p_1\}$; under Claim A, we have $\pi_j^* \notin S \cup T'$; by applying Claim A iteratively, and since $\pi_i \preceq_* \pi_j$, we conclude that $\pi_i \notin S \cup T'$, which is a contradiction because $\pi_i \in T'$.

Internal t^*-compatibility: Let i, j such that $\pi_i \preceq_* \pi_j$ and $\pi_i, \pi_j \in T'$. Since $T' \subset P$, we have $\pi_i, \pi_j \in P$. Hence, $i \leq j$ because π is internally t^*-compatible with P.

We now prove that (Ib) holds with T'. By definition of $y^{r\pi}_{p_1}$, both t^* and t' are optimal trees in $(N, P, y^{r\pi})$. Hence, $C(N) - y^{r\pi}(P) = C(S \cup T') - y^{r\pi}(T')$. Equivalently,

$$y^{r\pi}(P \setminus T') = C(N) - C(S \cup T'). \tag{10}$$

Let $\pi^b \in \Pi$ be an order t^*-compatible with $P \setminus T'$. We check that $y^{r\pi^b}(P \setminus T') = C(N) - C(S \cup T')$. By the induction hypothesis on statement (II), we can assume that nodes in $P \setminus T'$ follow the same order as in π. In particular, p_1 is still the first node in $P \setminus T'$ under π^b. Denote $P \setminus T' = \{p_1, \ldots, p_L\}$ following order π^b, i.e. when $p_i = \pi^b_{i^b}$ and $p_j = \pi^b_{j^b}$, then $i \leq j \Leftrightarrow i^b \leq j^b$. Internal t^*-compatibility assures that $p_i \preceq_* p_j \Rightarrow i \leq j$. For each $l \in \{1, \ldots, L\}$, let $G^l = \{i \in S \cup T' : i^* = p_l\}$ be the set of nodes in $S \cup T'$ that immediately follow p_l in t^* (case $G^l = \emptyset$ is also possible), and denote $G^l = \{g^l_1, \ldots, g^l_{k_l}\}$. We define the set of edges $E^l = \{e^l_1, \ldots, e^l_{k_l}\} \subset N^p$ inductively as follows: Let $e^l_1 = (i_{l1}, i'_{l1}) \in N^p$ be the first edge in the (unique) path in t' from g^l_1

to the source such that $i_{l1} \in F^{*g_1^l}$ and $i'_{l1} \notin F^{*g_1^l}$. Let $e_2^l = (i_{l2}, i'_{l2}) \in N^p$ be the first edge in the (unique) path in t' from g_2^l to the source such that $i_{l2} \in F^{*g_1^l} \cup F^{*g_2^l}$ and $i'_{l2} \notin F^{*g_1^l} \cup F^{*g_2^l}$. And so on. Thus, E^l is the set of nodes that connect each G^l to the source in t'. We now define the trees $t^0, t^1, \ldots, t^L$ inductively as $t^0 = t'$ and

$$t^l = \left(t^{l-1} \cup \{(i,i')\}_{i \in G^l \cup \{p_l\}}\right) \setminus E^l$$

for each $l = 1, \ldots, L$. By optimality of t^* and t', each t^l is also optimal in $(N, \{p_{l+1}, \ldots, p_L\}, y^{r\pi}, c)$. Hence, we have

$$\begin{aligned}
y_{p_L}^{r\pi} &= c\left(t^L\right) - c\left(t^{L-1}\right) = C(N) - C(N \setminus \{p_L\}) \\
y_{p_{L-1}}^{r\pi} &= c\left(t^{L-1}\right) - c\left(t^{L-2}\right) = C(N \setminus \{p_L\}) - C(N \setminus \{p_L, p_{L-1}\}) \\
&\vdots \\
y_{p_1}^{r\pi} &= c\left(t^1\right) - c\left(t^0\right) = C(S \cup T' \cup \{p_1\}) - C(S \cup T')
\end{aligned}$$

from where we deduce

$$\begin{aligned}
y_{p_L}^{r\pi^b} &= C(N) - C(N \setminus \{p_L\}) \\
y_{p_{L-1}}^{r\pi^b} &= C(N \setminus \{p_L\}) - C(N \setminus \{p_L, p_{L-1}\}) \\
&\vdots \\
y_{p_1}^{r\pi^b} &= C(S \cup T' \cup \{p_1\}) - C(S \cup T')
\end{aligned}$$

and, adding up these terms, we get

$$y^{r\pi^b}(P \setminus T') = C(N) - C(S \cup T') \tag{11}$$

so that (Ib) comes from (10) and (11).

We now prove statement (II), i.e. $y^{r\pi}(S) = \widehat{C}(S)$. Since $y^{r\pi} \in Core(C) = Core\left(\widehat{C}\right)$, we have $y^{r\pi}(S) \leq \widehat{C}(S)$. When $y^{r\pi}(S) = C(S)$, we have $\widehat{C}(S) \leq y^{r\pi}(S)$ because $\widehat{C}(S) \leq C(S)$. When $y^{r\pi}(S) \neq C(S)$, under statement (I) there exits $\emptyset \neq T' \subset P$ satisfying (Ia) and (Ib). We now apply the induction hypothesis on statement (II) to deduce that for all $\pi^a \in \Pi$ order t^*-compatible with T', and $\pi^b \in \Pi$ order t^*-compatible with $P \setminus T'$, we have $\widehat{C}(N \setminus T') = y^{r\pi^a}(N \setminus T')$ and $\widehat{C}(S \cup T') = y^{r\pi^b}(S \cup T')$. Under (I$a$) and (I$b$), we have

$y^{r\pi^a}(T') = y^{r\pi}(T')$ and $y^{r\pi^b}(P \setminus T') = y^{r\pi}(P \setminus T')$, respectively. Hence, $\widehat{C}(N \setminus T') = y^{r\pi}(N \setminus T')$ and $\widehat{C}(S \cup T') = y^{r\pi}(S \cup T')$. Thus,

$$\begin{aligned}
\widehat{C}(S) &= \min\left\{C(S), \min_{\emptyset \neq T'' \subset P}\left\{\widehat{C}(S \cup T'') + \widehat{C}\left(N \setminus T''\right) - C(N)\right\}\right\} \\
&\leq \min_{\emptyset \neq T'' \subset P}\left\{\widehat{C}\left(S \cup T''\right) + \widehat{C}\left(N \setminus T''\right) - C(N)\right\} \\
&\leq \widehat{C}(S \cup T') + \widehat{C}(N \setminus T') - C(N) \\
&= y^{r\pi}(S \cup T') + y^{r\pi}(N \setminus T') - C(N) \\
&= y^{r\pi}(S \cup T') + y^{r\pi}(N) - y^{r\pi}(T') - C(N) \\
&= y^{r\pi}(S).
\end{aligned}$$

concluding the proof. ■

We are now ready for the main result of this section. Let $\Delta(c)$ be the convex hull of the allocations $(y^{r\pi})_{\pi \in \Pi}$.

Theorem 3 *For any mcst game (N, C), $ExtCore(C) = \Delta(c)$.*

Proof. We proceed by contradiction. Assume $\Delta(c) \neq ExtCore(C)$. Since each $y \in \Delta(c)$ is a core allocation, we have that there exists an extreme core allocation x that does not belong to $\Delta(c)$. From this, we deduce that there exists some S such that $x(S) - y(S)$ has the same (nonzero) sign for each allocation $y \in \Delta(c)$. That is, either $x(S) - y(S) > 0$ for all $y \in \Delta(c)$, or $x(S) - y(S) < 0$ for all $y \in \Delta(c)$.

Then, S should be one of the sets that determine a saturate constraint on a face of $\Delta(c)$. Since $x \notin \Delta(c)$, then it should lay inside the opposite side of the half-space.

We can assume $x(S) - y(S) > 0$ for all $y \in \Delta$ because, in the opposite, we instead consider $T = N \setminus S$.

We now take y such that $y(S) = \widehat{C}(S)$. Existence of such a $y \in \Delta$ is guaranteed by Theorem 2. Then, $x(S) > \widehat{C}(S)$ and thus, by Lemma 1, x does not belong to $Core(\widehat{C}) = Core(C)$. Hence the contradiction. ■

As a corollary, we now obtain a complete description of the core.

Corollary 1 *For all $c \in \Gamma$, $Core(C) = \Delta(c)$.*

5 Link with cost sharing solutions

We show that while our method is new, it can be used to reinterpret the three most famous cost-sharing solutions in the literature on mcst.

A cost sharing solution (or rule) assigns a cost allocation $y(c)$ to any admissible cost matrix c. We start by building a solution from the allocations defined in the previous sections. Let

$$y^r = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^{r\pi}$$

be the average of the reduced marginal cost vectors. By our previous section, y^r is the barycenter of $Core(C)$.

5.1 The Bird solution

The Bird solution is defined as follows. Let $\Pi^*(c)$ be the set of permutations obtained in Prim's algorithm. For all $\pi \in \Pi^*(c)$, let $y^{b,\pi}_{\pi_i} = \min_{k=0,\ldots,i-1} c_{\pi_k \pi_i}$ where $\pi_0 = 0$. The Bird solution is

$$y^b = \frac{1}{|\Pi^*(c)|} \sum_{\pi \in \Pi^*(c)} y^{b,\pi}.$$

Theorem 4 *For all $\pi \in \Pi^*(c)$, $y^{b,\pi} = y^{r\pi}$.*

Proof. Suppose $\pi \in \Pi^*(c)$ and consider agent π_n. We have

$$y^{r\pi}_{\pi_n} = C(N) - C(N \setminus \{\pi_n\}) = \min_{k=0,\ldots,n-1} c_{\pi_k \pi_i} = y^{b,\pi}_{\pi_n}.$$

Observe that by construction, $C(S \cup \{\pi_n\}) - C(S) = \min_{k \in S} c_{\pi_k \pi_i} \leq C(N) - C(N \setminus \{\pi_n\})$ for all $S \subseteq N \setminus \pi_n$. Thus, $C^{\pi_n}(S) = \min(C(S), C(S \cup \pi_n) + C(N \setminus \pi_n) - C(N)) = C(S)$ for all $S \subseteq N \setminus \{\pi_n\}$.

Consider agent π_j for $j = 1, \ldots, n-1$.

$$\begin{aligned}
y^{r\pi}_{\pi_k} &= C^{\pi_n,\ldots,\pi_{j+1}}(\{\pi_1, \ldots, \pi_j\}) - C^{\pi_n,\ldots,\pi_{j+1}}(\{\pi_1, \ldots, \pi_{j-1}\}) \\
&= C(\{\pi_1, \ldots, \pi_j\}) - C(\{\pi_1, \ldots, \pi_{j-1}\}) \\
&= \min_{k=0,\ldots,j-1} c_{\pi_k \pi_j} \\
&= y^{b,\pi}_{\pi_j}
\end{aligned}$$

Hence, $y^{b,\pi} = y^{r\pi}$. ■

The Bird allocation is thus a very special case of our method, as it only uses the reduced marginal cost vector of C related to permutation π if π is a permutation corresponding to an order in which we construct the mcst using Prim's algorithm.

We thus obtain an alternative proof of the stability of the Bird allocation. In addition, we can see that the permutations used for the Bird allocation are such that there are no modifications to do on C before computing the marginal cost vector. For other permutations, we will obtain that the corresponding marginal cost vector (without modifying C) is not in the core.

5.2 The cycle-complete solution

To define the cycle-complete solution, we need to define irreducible and cycle-complete cost matrices.

From any cost matrix c, we can define the irreducible cost matrix $\bar{c}$ as follows:

$$\bar{c}_{ij} = \min_{p_{ij} \in P_{ij}(N_0)} \max_{e \in p_{ij}} c_e \text{ for all } i, j \in N_0.$$

From any cost matrix c, we can define the cycle-complete cost matrix c^* as follows:

$$\begin{aligned} c^*_{ij} &= \max_{k \in N \setminus \{i,j\}} \overline{c_{ij}^{N\setminus\{k\}}} \text{ for all } i, j \in N \\ c^*_{0i} &= \max_{k \in N \setminus \{i\}} \overline{c_{ij}^{N\setminus\{k\}}} \text{ for all } i \in N. \end{aligned}$$

where $\overline{c_{ij}^{N\setminus\{k\}}}$ indicate the cost of edge (i, j) on the matrix that we first restricted to agents in $N \setminus \{k\}$ before transforming into an irreducible matrix.

The cycle complete matrix can also be defined using cycles (Trudeau (2012)): for edge (i, j), we look at cycles that go through i and j. If there is one such cycle such that its most expensive edge is cheaper than a direct connection on edge (i, j), we assign this cost to edge (i, j).

The cycle-complete solution y^{cc} is the Shapley value of (N, C).

Let Γ^e be the set of elementary cost matrices: for any $c \in \Gamma^e$ and any $i, j \in N_0$, $c_{ij} \in \{0, 1\}$. We show that for elementary cost matrices, the cycle-complete solution corresponds to y^r.

Theorem 5 *For any elementary cost matrix $c \in \Gamma^e$, $y^r = y^{cc}$.*

Proof. Trudeau (2012) showed that for elementary problems, $Core(C) = Core(C^*)$, where (N, C^*) is the mcst game associated to c^*, and that (N, C^*) is concave. By the properties of the Shapley value for concave games, y^{cc} is the average over the set of permutations of incremental cost allocations, with each of them being an extreme core allocation. It is obvious that the incremental cost vector corresponding to order π is exactly $y^{r\pi}$. We thus have that $y^r = y^{cc}$. ■

An alternative explanation of the above result is that the changes from the original to the cycle-complete cost matrix are the same as those imposed by our method. If node i has two free distinct paths to node j, say with the help of S and T, she will obtain the cost savings with both coalitions. This will result in $\widehat{C}(\{i,j\}) = 0$, the same result as if we modified directly the matrix into a cycle-complete matrix.

5.3 The folk solution

The folk solution is the Shapley value of the cost game associated with the irreducible cost matrix $\bar{c}$ defined in the previous subsection. As for the cycle-complete solution, we show a link between our method and the folk solution in elementary mcst problems. To do so, we need to define the monotonically increasing version of C, denoted as C^+:

Definition 2 *For all $c \in \Gamma$, let (N, C^+) be defined in the following way:*
Step 0: $C^+(N) = C(N)$.
Step k: For all S such that $|S| = n - k$,

$$C^+(S,c) = \min(C(S,c), \min_{i \in N \setminus S} C^+(S \cup i), c)$$

for $0 < k < n$.
Step n: $C^+(\emptyset) = 0$.

The transformation from C to C^+ guarantees that there is never a negative cost to add an agent to a coalition. We show that applying our method to C^+ for an elementary problem yields the folk solution.

Theorem 6 *For any elementary cost matrix $c \in \Gamma^e$, $y^r(C^+) = y^f$, where $y^r(C^+)$ is y^r defined from C^+ instead of C.*

Proof. Consider a connected component $T \subseteq N_0$. If $0 \in T$, then $C(T \setminus \{0\}) = 0$. Otherwise, $C(T \setminus \{0\}) = 1$. Suppose that for $S \in T \setminus \{0\}$, $C(S \setminus \{0\}) > C(T \setminus \{0\})$. In such a case, $\bar{c}$ is such that $\bar{c}_{ij} = 0$ for all $i, j \in T$. By monotonicity, we must have that $C(S \setminus \{0\}) = C(T \setminus \{0\})$ for all $S \subseteq T$. Therefore, in both cases, the changes are identical.

We next show that there are no other changes. If i, j belong to different connected components, say T_1 and T_2, then $\bar{c}_{ij} = c_{ij}$. We also have that for all $R \subseteq T_1$ and $S \subseteq T_2$, $C(R \cup S) = C(R) = C(S)$. Therefore, once we have made the cost function monotonic within the connected components, it will be monotonic over all coalitions, meaning that we do not need to make any other changes.

After the modifications, to obtain y^f we take the Shapley value of $(N, \bar{C})$, the cost game associated with $\bar{c}$, or equivalently, $\sum_{\pi \in \Pi(N)} \frac{1}{n!} y^\pi(\bar{C})$. Given that $(N, \bar{C})$ is concave (Bergantiños and Vidal-Puga (2007)), $y^{r\pi}(\bar{C}) = y^\pi(\bar{C})$ for all $\pi \in \Pi(N)$ and thus $y^r(C^+) = y^f$. ∎

We thus obtain the new result that for elementary mcst problems, the folk solution is the barycenter of $Core(C^+)$. Since in elementary mcst problems moving from $Core(C)$ to $Core(C^+)$ is the same as restricting our attention to non-negative core allocations, we can thus say that for elementary mcst problems, the folk solution is the barycenter of the non-negative core.

Therefore, we obtain, for elementary cost matrices, a clear distinction between the folk and cycle-complete solutions, based on whether or not they disqualify core allocations where some agents receive strictly negative cost shares.

6 Discussion

The result of the previous section on the folk and cycle-complete solutions do not hold for non-elementary cost matrices. For those, we can compute the folk and cycle-complete solutions by decomposing the cost matrix into a series of elementary cost matrices and summing up. While that approach is computationally advantageous, one of the disadvantage is that, in general, we have that $Core(C^*)$ is a strict subset of $Core(C)$ and $Core(\bar{C})$ is a strict subset of $Core(C^+)$; the cycle-complete and folk solutions are no longer the barycenters of, respectively, the core of C and the non-negative core of C.

If we are willing to forego the Piecewise Linearity property, we therefore can use y^r and $y^r(C^+)$ as non-piecewise linear extensions of, respectively, the cycle-complete and folk solutions.

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